QUASILINEAR SYSTEMS OF NEUTRAL TYPE WITH A DEVIATING ARGUMENT

(KVAZILINNEINYE SISTEMY S OTKLONIAIUSHCHIMSIA ARGUMENTOM NEITRAL'NOGO TIPA)

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A.M. RODIONOV
(Moscow)

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We consider quasilinear systems of neutral type with constant coefficients and with constant positive deviations

$$\frac{dx(t)}{dt}+\sum_{p=0}^{1}\sum_{j=1}^{q}a_{pj}x^{(p)}(t-\tau_j)=f(t)+\mu F(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_q), \mu)$$

or, briefly

$$Ux=f+\mu F \tag{1}$$

Here μ is a small parameter, and

$$x(t)=\{x_1(t), \ldots, x_n(t)\}, \qquad f(t)=\{f_1(t), \ldots, f_n(t)\}, \qquad F=\{F_1, \ldots, F_n\}$$

$$a_{pj}=\|a_{pjsk}\| \qquad (s, k=1, \ldots, n)$$

The function F is continuous in t and has a period 2π; with a constant small μ and with values of $x(t)$, $x(t-\tau_j)$ lying within a certain region G of the domain of these variables, it has continuous partial derivatives of the first order; the function $f(t)$ is a continuous periodic function of period 2π.

With the aid of the principle of compressed mapping, we demonstrate the existence of a unique periodic solution, which for $\mu = 0$ reduces to the solution of the corresponding generating equation. Nonresonant, resonant and "exceptional" cases are represented [1].

1. Nonresonant case. We assume that no integral frequencies exist for the characteristic equation

$$\Delta(\lambda)=\left|\lambda\left(E+\sum_{j=1}^{q}a_{1j}e^{-\lambda\tau_j}\right)+\sum_{j=1}^{q}a_{0j}e^{-\lambda\tau_j}\right|=0 \tag{2}$$

corresponding to Equation (1).

We shall consider a closed subset $C_p(0 \leqslant t \leqslant 2\pi)$ of such functions for which $x^{(i)}(0) = x^{(i)}(2\pi)(i = 0, 1, \ldots, p)$; these functions are located inside the interval $(0, 2\pi)$ such that they are periodic with a period 2π. In this sense we shall speak of periodic functions in the domain $C_p(-\infty < t < +\infty)$.

We consider the generating system

$$Ux = f \tag{3}$$

The left-hand side of (3) determines the operator U, having meaning in every case for any function x in C_1. The value of U is an arbitrary function in C.

It is easy to prove that U is a linear (i.e. additive and continuous) operator. Since (2) has no integral frequencies, then $Ux = \theta$ only for $x = \theta$.

Indeed, the operator U exists and establishes a one-to-one correspondence between the complete domains C_1 and C. In agreement with the theorem of Banakh [2] the inverse operator U^{-1} is linear. It follows that

$$\| U^{-1}f \| \leqslant C \| f \|$$

Consequently, for a single periodic solution of the system (3) we have the estimate

$$| x_s^{\circ}(t) | \leqslant CM \tag{4}$$

where M is determined by the condition $| f_s | \leqslant M$. By use of the estimate (4) we apply the principle of compressed mapping for sufficiently small μ to the equation

$$Ux^{(l)} = f(t) + \mu F(t, x^{(l-1)}(t), x^{(l-1)}(t - \tau_j), \mu)$$

in a class of functions reducing for $\mu = 0$ to the solution $x^{\circ}(t)$ of the generating system (3) (supposing that $x^{\circ}(t) \in G$). From this, the theorem follows.

Theorem 1. The system (1) in the nonresonant case has, for a sufficiently small μ, a unique periodic solution of period 2π, reducing for $\mu = 0$ to the solution of the generating system (3).

2. *Resonant case.* Let Equation (2) have a finite number of integral frequencies and let $\phi_m = C_m \exp iN_m (m = 1, \ldots, r)$ corresponding to the periodic solutions of the homogeneous system $Ux = \theta$. In this case a

periodic solution does not ordinarily exist and the function $f(t)$ must satisfy certain conditions.

Along with the system (3) we consider the conjugate system. According to the theorem of Riss on the domain C, the linear functional is

$$g(f)=\int_0^{2\pi}(f,\ d\psi),\qquad (f,d\psi)=\sum_{s=1}^{n} f_s(t)\,d\psi_s(t)$$

where $\psi \in V_0$ is a periodic function of t with period 2π (V_0 denotes the set of all regular functions with $\psi(0)=0$), $(f,\ d\psi)$ is a scalar product of the vector functions f and $d\psi$. For determination of the conjugate system we have $U^*g = g(Ux)$. We obtain, after simple calculations

$$U^*g = g(Ux)$$

$$=\int_0^{2\pi}\sum_{s=1}^{n} x_s(t)\,d\left\{-\frac{d\psi_s(t)}{dt}-\sum_{j=1}^{q}\sum_{k=1}^{n} a_{1jks}\frac{d\psi_s(t+\tau_j)}{dt}+\sum_{j=1}^{q}\sum_{k=1}^{n} a_{0jks}\,\psi_s(t+\tau_j)\right\}$$

Consequently, $U^*g = \theta$ on the functionals with $\psi_s(t) = \phi_{sm}^*(t) - \phi_{sm}^*(0)$, where ϕ_{sm}^* is a periodic solution of the system

$$\frac{d\varphi_m^*(t)}{dt}+\sum_{j=1}^{q} a_{1j}^*\frac{d\varphi_m^*(t+\tau_j)}{dt}-\sum_{j=1}^{q} a_{0j}^*\varphi_m^*(t+\tau_j)=0\qquad (m=1,\ldots,r)\qquad (5)$$

Here a_{1j}^* and a_{0j}^* are transposed matrices (the fundamental equation corresponding to (5) has roots differing in sign from those for Equation (2)).

Banakh established the theorem that the system $Ux = f$ can be solved only when $g(f) = 0$ follows from $U^*g = \theta$.

Indeed, for the system (3) to have a periodic solution in the resonant case the function $f(t)$ must satisfy the following r conditions:

$$\int_0^{2\pi}\sum_{s=1}^{n} f_s(t)\,\varphi_{sm}^*(t)\,dt=0\qquad (m=1,\ldots,r)\qquad (6)$$

Functions $f(t)$ satisfying the conditions (6) evidently generate a linear and closed set which we call N. Consequently, we have $U(C_1) = N$. The solution of the system (3) may be written in the form

$$x_s^\circ(t)=\sum_{m=1}^{r} M_m^\circ\varphi_{sm}(t)+X_s(t)\qquad (|X_s(t)|\leqslant C^*M)$$

Here $X_s(t)$ is a particular solution of the homogeneous system (3), C^* is a constant independent of the form of $f(t)$, since according to the previously mentioned Banakh theorem the operator U is a homomorphism of the domain C_1 on N.

The method of successive approximations leads in the case of resonance to the following expressions:

$$Ux^{(l)} = f(t) + \mu F(t, x^{(l-1)}(t), x^{(l-1)}(t-\tau_j), \mu) \tag{7}$$

$$x_s^{(l)} = \sum_{m=1}^{r} M_m^{(l)} \varphi_{sm}(t) + x_s^{\circ}(t) + \mu L_s(F^{(l-1)}) \tag{8}$$

Here $F^{(l-1)} = F(t, x^{(l-1)}(t), x^{(l-1)}(t-\tau_j), \mu)$, and $L_s(F^{(l-1)})$ represents a particular solution of the system (7) corresponding to a known function $F^{(l-1)}$ of time; the coefficients $M_m^{(l)}$ are determined from the condition of the absence of resonating terms on the right-hand side of the system

$$P_m(M_1^{(l)}, \ldots, M_r^{(l)}, \mu) = \int_0^{2\pi} (F^{(l)}, \varphi_m^*)\, dt = 0 \qquad (m = 1, \ldots, r) \tag{9}$$

If in the class of functions reducing to the solution of the generating system with $\mu = 0$ there exists a solution to the basic system (1), then for $\mu = 0$ all $x^{(l)}(t)$ must reduce to $x^{\circ}(t)$ in the successive approximations to the exact solution $x(t, \mu)$. As is evident from (8) and (9), this requires that the constants M_1°, ..., M_r° be determined from the equation

$$P_m^{\circ}(M_1^{\circ}, \ldots, M_r^{\circ}, 0) = \int_0^{2\pi} \sum_{s=1}^{n} F_s(t, x^{\circ}(t), x^{\circ}(t-\tau_j), 0)\, \varphi_{sm}^*(t)\, dt = 0 \tag{10}$$

For this, the condition

$$\frac{\partial(P_1^{\circ}, \ldots, P_r^{\circ})}{\partial(M_1^{\circ}, \ldots, M_r^{\circ})} \neq 0 \tag{11}$$

must be fulfilled.

One may then show that application of the principle of compressed mapping is similar to that in the work of Malkin [3]. The theorem follows.

Theorem 2. In order that the system (1) have a solution $x(t, \mu)$, reducing for $\mu = 0$ to the solution of the generating system $x^{\circ}(t)$, it is necessary that a unique periodic solution of (1) exist, reducing for $\mu = 0$ to the solution of the generating system, with condition (11) being satisfied to assure fulfilment of condition (10).

Note 1. The problem of finding the periodic solutions of automatic systems is similar to that in the case of ordinary differential equations.

Note 2. Special procedure is required for the exceptional case where Equation (2) has a countable set of integral natural frequencies [1]. We remark [4] that this case is realized when

$$\| a_{1jsk} \| = 0 \qquad \begin{pmatrix} j = 1, \ldots, q \\ s, k = 1, \ldots, n \end{pmatrix}$$

i.e. Equation (1) represents a system with delay.

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